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On the Convergence to a Statistical Equilibrium for the Dirac Equation

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Abstract

We consider the Dirac equation in $\mathbb{R}^3$ with constant coefficients and study the distribution μ_t of the random solution at time $t \in \mathbb{R}$. It is assumed that the initial measure μ_0 has zero mean, a translation-invariant covariance, and finite mean charge density. We also assume that μ_0 satisfies a mixing condition of Rosenblatt- or Ibragimov-Linnik-type. The main result is the convergence of μ_t to a Gaussian measure as $t \to \infty$. The proof uses the study of long time asymptotics of the solution and S.N. Bernstein's "room-corridor" method.

Key words and phrases: Dirac equation, random initial data, mixing condition, Gaussian measures, covariance matrices, characteristic functional

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1 Introduction

This paper can be regarded as a continuation of our papers [1]-[5] concerning the analysis of long-time convergence to an equilibrium distribution for hyperbolic partial differential equations and harmonic crystals. Here we develop the analysis for the Dirac equation

$$\left\{ \begin{array}{l} \dot\psi(x,t)=[-\alpha\cdot\nabla-i\beta m]\,\psi(x,t), \quad x\in\mathbb{R}^3, \\ \psi(x,0)=\psi_0(x), \end{array} \right. \tag{1.1}$$

where $\nabla=(\partial_1,\partial_2,\partial_3)$, $\partial_k=\partial/\partial x_k$, $m>0$, $\alpha=(\alpha_1,\alpha_2,\alpha_3)$, α_k and β are 4×4 Dirac matrices (see (2.12), (2.13)). The solution $\psi(x,t)\in\mathbb{C}^4$ for $(x,t)\in\mathbb{R}^4$.

It is assumed that the initial data $\psi_0(x)$ are given by a random element of the function space $\mathcal{H}\equiv H^0_{loc}(\mathbb{R}^3)$ of states with finite local energy, see Definition 2.1 below. The distribution of ψ_0 is a zero-mean probability measure μ_0 satisfying some additional assumptions, see Conditions **S1-S3** below. Denote by μ_t, $t\in\mathbb{R}$, the measure on $\mathcal{H}$ giving the distribution of the random solution $\psi(t)$ of problem (1.1). We identify the complex and real spaces $\mathbb{C}^4\equiv\mathbb{R}^8$, and $\otimes$ stands for the tensor product of real vectors. The correlation functions of the initial measure are supposed to be translation-invariant,

$$Q_0(x,y):=E\Big(\psi_0(x)\otimes\psi_0(y)\Big)=q_0(x-y),\ \ x,y\in\mathbb{R}^3. \tag{1.2}$$

We also assume that the initial mean charge density is finite,

$$e_0:=E|\psi_0(x)|^2=\operatorname{tr}q_0(0)<\infty, \quad x\in\mathbb{R}^3. \tag{1.3}$$

Finally, assume that the measure μ_0 satisfies a mixing condition of a Rosenblatt- or Ibragimov-Linnik type, which means that

$$\psi_0(x)\ \ \textit{and}\ \ \psi_0(y)\ \ \textit{are}\ \ \textit{asymptotically}\ \ \textit{independent}\ \ \textit{as}\ \ |x-y|\to\infty. \tag{1.4}$$

Our main result gives the (weak) convergence of μ_t to a limit measure μ_∞,

$$\mu_t \rightharpoondown \mu_\infty, \quad t\to\infty, \tag{1.5}$$

which is an equilibrium Gaussian measure on $\mathcal{H}$. A similar convergence holds for $t\to-\infty$ because our system is time-reversible. Explicit formulas (2.17) for the correlation functions of μ_∞ are given.

To prove the convergence (1.5) we follow the strategy of [1]-[5]. There are three steps.
I. The family of measures μ_t, $t\ge 0$, is weakly compact in an appropriate Fréchet space.
II. The correlation functions converge to a limit,

$$Q_t(x,y)\equiv\int\psi(x)\otimes\psi(y)\,\mu_t(\psi)\to Q_\infty(x,y), \quad t\to\infty. \tag{1.6}$$

III. The characteristic functionals converge to a Gaussian functional,

$$\hat\mu_t(\phi):=\int e^{i\langle\psi,\phi\rangle}\mu_t(d\psi)\to\exp\{-\frac{1}{2}\mathcal{Q}_\infty(\phi,\phi)\}, \quad t\to\infty. \tag{1.7}$$

Here $\langle\cdot,\cdot\rangle$ stands for a real scalar product in L^2, $\mathcal{Q}_\infty$ for a quadratic form with the integral kernel $Q_\infty(x,y)$, and ϕ for an arbitrary element of the dual space.

Property **I** follows from the Prokhorov Compactness Theorem by a method used in [13]. Namely, we first establish a uniform bound for the mean local charge with respect to the measure μ_t, $t \ge 0$. Then the Prokhorov condition follows from the Sobolev embedding theorem by Chebyshev's inequality. Property **II** is derived from an analysis of oscillatory integrals arising in the Fourier transform. However, the Fourier transform by itself is insufficient to prove Property **III**. We derive it by using an explicit representation of the solution in the coordinate space with the help of the Bernstein' "room-corridor" technique by a method of [1]-[5]. The method gives a representation of the solution as a sum of weakly dependent random variables. Then (1.5) follows from the Ibragimov-Linnik central limit theorem under a Lindeberg-type condition. We sketch the proofs by using the technique of [1].

The paper is organized as follows. The main result is stated in Section 2. The compactness (Property **I**) is established in Section 3, the convergence (1.6) in Section 4, and the convergence (1.7) in Sections 5.

2 Main results

Let us describe our results more precisely.

2.1 Notation

We assume that the initial date ψ_0 in (1.1) is complex-valued vector function belonging to the phase space $\mathcal{H}$.

Definition 2.1 *Denote by $\mathcal{H} \equiv H^0_{\rm loc}(\mathbb{R}^3, \mathbb{C}^4)$ the Fréchet space of complex-valued functions $\psi(x)$, endowed with local energy seminorms*

$$\|\psi\|^2_{0,R} \equiv \int\limits_{|x|<R} |\psi(x)|^2\, dx < \infty, \quad \forall R > 0. \tag{2.1}$$

Proposition 2.2 *(i) For any $\psi_0 \in \mathcal{H}$ there exists a unique solution $\psi(\cdot, t) \in C(\mathbb{R}, \mathcal{H})$ to Cauchy problem (1.1).*
(ii) For any $t \in \mathbb{R}$, the operator $U(t) : \psi_0 \mapsto \psi(\cdot, t)$ is continuous in $\mathcal{H}$.

Proposition 2.2 follows from [10, Thms. V.3.1, V.3.2]) because the speed of propagation for Eq. (1.1) is finite.

Let us choose a function $\zeta(x) \in C_0^\infty(\mathbb{R}^3)$ such that $\zeta(0) \ne 0$. Denote by $H^s_{\rm loc}(\mathbb{R}^3)$, $s \in \mathbb{R}$, the local Sobolev spaces, i.e., the Fréchet spaces of distributions $u \in D'(\mathbb{R}^3)$ with the finite seminorms

$$\|u\|_{s,R} := \|\Lambda^s\Big(\zeta(x/R)u\Big)\|_{L^2(\mathbb{R}^3)}, \tag{2.2}$$

where $\Lambda^s v := F^{-1}_{k\to x}(\langle k\rangle^s \hat{v}(k))$, $\langle k\rangle := \sqrt{|k|^2+1}$, and $\hat{v} := Fv$ is the Fourier transform of a tempered distribution v. For $\phi \in C_0^\infty(\mathbb{R}^3)$ write $F\phi(k) = \int e^{ik\cdot x}\phi(x)dx$. Note that the space $H^s_{\rm loc}(\mathbb{R}^3)$ for $s = 0$ agrees with Definition 2.1.

Definition 2.3 *For $s \in \mathbb{R}$, write $\mathcal{H}^s \equiv H^s_{\rm loc}(\mathbb{R}^3)$.*

Using the standard technique of pseudodifferential operators and Sobolev's embedding theorem (see, e.g., [8]), one can prove that $\mathcal{H} = \mathcal{H}^0 \subset \mathcal{H}^{-\varepsilon}$ for every $\varepsilon > 0$, and the embedding is compact.

2.2 Random solution. Convergence to equilibrium

Let (Ω, Σ, P) be a probability space with expectation E and let $\mathcal{B}(\mathcal{H})$ be the Borel σ-algebra of $\mathcal{H}$. Assume that $\psi_0 = \psi_0(\omega, \cdot)$ in (1.1) is a measurable random function with values in $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$. In other words, $(\omega, x) \mapsto \psi_0(\omega, x)$ is a measurable mapping $\Omega \times \mathbb{R}^3 \to \mathbb{C}^4$ with respect to the (completed) σ-algebras $\Sigma \times \mathcal{B}(\mathbb{R}^3)$ and $\mathcal{B}(\mathbb{C}^4)$. Then, by virtue of Proposition 2.2, $\psi(t) = U(t)\psi_0$ is again a measurable random function with values in $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$. Denote by $\mu_0(d\psi_0)$ the Borel probability measure on $\mathcal{H}$ giving the distribution of ψ_0. Without loss of generality, we can assume that $(\Omega, \Sigma, P) = (\mathcal{H}, \mathcal{B}(\mathcal{H}), \mu_0)$ and $\psi_0(\omega, x) = \omega(x)$ for $\mu_0(d\omega) \times dx$-almost all points $(\omega, x) \in \mathcal{H} \times \mathbb{R}^3$.

Definition 2.4 *Let μ_t be the probability measure on $\mathcal{H}$ giving the distribution of $Y(t)$,*

$$\mu_t(B) = \mu_0(U(-t)B), \quad \forall B \in \mathcal{B}(\mathcal{H}), \ \ t \in \mathbb{R}. \tag{2.3}$$

Our main objective is to derive the weak convergence of the measures μ_t in the Fréchet space $\mathcal{H}^{-\varepsilon}$ for each $\varepsilon > 0$,

$$\mu_t \overset{\mathcal{H}^{-\varepsilon}}{\longrightarrow} \mu_\infty \ \text{ as } \ t \to \infty, \tag{2.4}$$

where μ_∞ is some Borel probability measure on the space $\mathcal{H}$. This means the convergence

$$\int f(\psi)\mu_t(d\psi) \to \int f(\psi)\mu_\infty(d\psi), \qquad t \to \infty, \tag{2.5}$$

for any bounded continuous functional $f(\psi)$ on $\mathcal{H}^{-\varepsilon}$.

Set $\mathcal{R}\psi \equiv (\operatorname{Re}\psi, \operatorname{Im}\psi) = \{\operatorname{Re}\psi_1, \ldots, \operatorname{Re}\psi_4, \operatorname{Im}\psi_1, \ldots, \operatorname{Im}\psi_4\}$ for $\psi = (\psi_1, \ldots \psi_4) \in \mathbb{C}^4$, and denote by $\mathcal{R}^j\psi$ j th component of the vector $\mathcal{R}\psi$, $j = 1, ..., 8$. The brackets $(\cdot, \cdot)$ mean the inner product in the real Hilbert spaces $L^2 \equiv L^2(\mathbb{R}^3)$, in $L^2 \otimes \mathbb{R}^N$, or in some their extensions. For $\psi(x), \phi(x) \in L^2(\mathbb{R}^3, \mathbb{C}^4)$, write

$$\langle \psi, \phi \rangle := (\mathcal{R}\psi, \mathcal{R}\phi) = \sum_{j=1}^{8} (\mathcal{R}^j\psi, \mathcal{R}^j\phi). \tag{2.6}$$

Definition 2.5 *The correlation functions of the measure μ_t are defined by*

$$Q_t^{ij}(x, y) \equiv E\Big(\mathcal{R}^i\psi(x)\mathcal{R}^j\psi(y)\Big) \quad \textit{for almost all } \ x, y \in \mathbb{R}^3, \quad i, j = 1, ..., 8, \tag{2.7}$$

provided that the expectations in the RHS are finite.

Denote by D the space of complex-valued functions in $C_0^\infty(\mathbb{R}^3)$ and write $\mathcal{D} := [D]^4$. For a Borel probability measure μ on $\mathcal{H}$, denote by $\hat\mu$ the characteristic functional (the Fourier transform)

$$\hat\mu(\phi) \equiv \int \exp(i\langle \psi, \phi \rangle)\, \mu(d\psi), \ \ \phi \in \mathcal{D} \quad (\text{see } \ (2.6)).$$

A measure μ is said to be *Gaussian* (with zero expectation) if its characteristic functional is of the form

$$\hat\mu(\phi) = \exp\Big\{ -\frac{1}{2}\mathcal{Q}(\phi, \phi) \Big\}, \ \ \phi \in \mathcal{D},$$

where $\mathcal{Q}$ is a real nonnegative quadratic form on $\mathcal{D}$. A measure μ is said to be *translation-invariant* if

$$\mu(T_h B) = \mu(B), \quad B \in \mathcal{B}(\mathcal{H}), \quad h \in \mathbb{R}^3,$$

where $T_h\psi(x) = \psi(x - h)$, $x \in \mathbb{R}^3$.

2.3 Mixing condition

Let $O(r)$ be the set of all pairs of open bounded subsets $\mathcal{A}, \mathcal{B} \subset \mathbb{R}^3$ at the distance $\operatorname{dist}(\mathcal{A}, \mathcal{B}) \geq r$, and let $\sigma(\mathcal{A})$ be the σ-algebra in $\mathcal{H}$ generated by the linear functionals $\psi \mapsto \langle \psi, \phi \rangle$, where $\phi \in \mathcal{D}$ with $\operatorname{supp} \phi \subset \mathcal{A}$. Define the Ibragimov-Linnik mixing coefficient of a probability measure μ_0 on $\mathcal{H}$ by formula (cf. [9, Def. 17.2.2])

$$\varphi(r) \equiv \sup_{(\mathcal{A},\mathcal{B}) \in O(r)} \sup_{\substack{A \in \sigma(\mathcal{A}), B \in \sigma(\mathcal{B}) \\ \mu_0(B) > 0}} \frac{|\mu_0(A \cap B) - \mu_0(A)\mu_0(B)|}{\mu_0(B)}. \tag{2.8}$$

Definition 2.6 *We say that the measure μ_0 satisfies the strong, uniform Ibragimov-Linnik mixing condition if*

$$\varphi(r) \to 0 \quad \text{as} \quad r \to \infty. \tag{2.9}$$

We specify the rate of decay of φ below (see Condition **S3**).

2.4 Main assumptions and results

We assume that the measure μ_0 has the following properties **S0–S3**:

S0. μ_0 has zero expectation value, $E\psi_0(x) \equiv 0$, $x \in \mathbb{R}^3$.
S1. μ_0 has translation-invariant correlation functions,

$$Q_0^{ij}(x, y) \equiv E\Big(\mathcal{R}^i\psi(x)\mathcal{R}^j\psi(y)\Big) = q_0^{ij}(x-y) \text{ for almost all } x, y \in \mathbb{R}^3, \ i, j = 1, ..., 8. \tag{2.10}$$

S2. μ_0 has a finite mean charge density, i.e., Eq. (1.3) holds.
S3. μ_0 satisfies the strong uniform Ibragimov-Linnik mixing condition, with

$$\int_0^\infty r^2 \varphi^{1/2}(r) dr < \infty. \tag{2.11}$$

The standard form of the Dirac matrices α_k and β (in 2×2 blocks) is

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \quad (k = 1, 2, 3), \tag{2.12}$$

where I denotes the unit matrix and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.13}$$

Introduce the following 8×8 real valued matrices (in 4×4 blocks)

$$\Lambda_1 = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_1 \end{pmatrix}, \ \Lambda_2 = \begin{pmatrix} 0 & i\alpha_2 \\ -i\alpha_2 & 0 \end{pmatrix}, \ \Lambda_3 = \begin{pmatrix} \alpha_3 & 0 \\ 0 & \alpha_3 \end{pmatrix}, \ \Lambda_0 = \begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix}. \tag{2.14}$$

Note that by (2.12) and (2.13) we have

$$i\alpha_2 = \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}, \quad \text{where } i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Moreover, $\Lambda_k^T=\Lambda_k$, $k=1,2,3$, $\Lambda_0^T=-\Lambda_0$. Write

$$\Lambda=(\Lambda_1,\Lambda_2,\Lambda_3),\quad P(\nabla)=\Lambda\cdot\nabla+m\Lambda_0. \tag{2.15}$$

For almost all $x,y\in\mathbb{R}^3$, introduce the matrix-valued function

$$Q_\infty(x,y)\equiv\Big(Q_\infty^{ij}(x,y)\Big)_{i,j=1,\dots,8}=\Big(q_\infty^{ij}(x-y)\Big)_{i,j=1,\dots,8}. \tag{2.16}$$

Here

$$\hat q_\infty(k)=\frac12\hat q_0(k)+\frac12\hat{\mathcal{P}}(k)P(-ik)\hat q_0(k)P^T(ik), \tag{2.17}$$

where $\hat{\mathcal{P}}(k)=1/(k^2+m^2)$, and $\hat q_0(k)$ is the Fourier transform of the correlation matrix of the measure μ_0 (see (2.10)). Since $P^T(ik)=-P(-ik)$, we have, formally,

$$q_\infty(z)=\frac12 q_0(z)-\frac12\mathcal{P}*P(\nabla)q_0(z)P(\overleftarrow{\nabla}), \tag{2.18}$$

where $\mathcal{P}(z)=e^{-m|z|}/(4\pi|z|)$ is the fundamental solution for the operator $-\Delta+m^2$, and $*$ stands fors the convolution of distributions. We show below that $\hat q_0\in L^2\equiv L^2(\mathbb{R}^3)$ (cf (4.7)). Hence, $\hat q_\infty(k)\in L^2$ by (2.17), and the convolution in (2.18) also belongs to L^2 .

Denote by $\mathcal{Q}_\infty$ a real quadratic form on L^2 defined by

$$\mathcal{Q}_\infty(\phi,\phi)\equiv(Q_\infty(x,y),\mathcal{R}\phi(x)\otimes\mathcal{R}\phi(y))=\sum_{i,j=1}^{8}\int\limits_{\mathbb{R}^3\times\mathbb{R}^3}Q_\infty^{ij}(x,y)\mathcal{R}^i\phi(x)\mathcal{R}^j\phi(y)\,dxdy.$$

The form $\mathcal{Q}_\infty$ is continuous in L^2 because $\hat q_\infty(k)$ is bounded by Corollary 4.3.

Theorem A. *Let $m>0$, and let* **S0–S3** *hold. Then*
(i) the convergence in (2.4) holds for any $\varepsilon>0$.
(ii) The limit measure μ_∞ is a Gaussian equilibrium measure on $\mathcal{H}$.
(iii) The characteristic functional of μ_∞ is of the form

$$\hat\mu_\infty(\phi)=\exp\Big\{-\frac12\mathcal{Q}_\infty(\phi,\phi)\Big\},\ \ \phi\in\mathcal{D}.$$

Theorem A can be derived from Propositions 2.7 and 2.8 given below by using the same arguments as in [13, Theorem XII.5.2].

Proposition 2.7 *The family of measures $\{\mu_t,t\in\mathbb{R}\}$ is weakly compact in the space $\mathcal{H}^{-\varepsilon}$ for any $\varepsilon>0$.*

Proposition 2.8 *For any $\phi\in\mathcal{D}$,*

$$\hat\mu_t(\phi)\equiv\int\exp\{i\langle\psi,\phi\rangle\}\,\mu_t(d\psi)=E\exp\{i\langle U(t)\psi,\phi\rangle\}\to\exp\Big\{-\frac12\mathcal{Q}_\infty(\phi,\phi)\Big\},\ t\to\infty \tag{2.19}$$

Propositions 2.7 and 2.8 are proved in Sections 3 and 4-5, respectively.

2.5 Remark on various mixing conditions for the initial measure

We use the strong uniform Ibragimov-Linnik mixing condition for the simplicity of our presentation. The *uniform* Rosenblatt mixing condition [12] with a higher degree >2 in the bound (1.3) is also sufficient. In this case we assume that there exists a δ, $\delta>0$, such that $\sup_{x\in\mathbb{R}^3} E|\psi_0(x)|^{2+\delta}<\infty$. Then condition (2.11) requires the following modification:

$$\int_0^\infty r\alpha^p(r)dr<\infty, \quad p=\min(\delta/(2+\delta),1/2),$$

where $\alpha(r)$ is the Rosenblatt mixing coefficient defined as in (2.8), but without the denominator $\mu_0(B)$. The statements of Theorem A and their proofs remain essentially unchanged.

3 Compactness of measures

3.1 Fundamental solution of the Dirac operator

One can easily check that α_k and β are Hermitian symmetric matrices satisfying the anticommutation relations

$$\begin{cases} \alpha_k^*=\alpha_k, & \beta^*=\beta, \\ \alpha_k\alpha_l+\alpha_l\alpha_k=2\delta_{kl}I, & \alpha_k\beta+\beta\alpha_k=0, \end{cases}$$

(δ_{kl} is Kronecker's delta). Therefore,

$$(\partial_t+\alpha\cdot\nabla+i\beta m)\,(\partial_t-\alpha\cdot\nabla-i\beta m)=(\partial_t^2-\triangle+m^2)I.$$

Then we can construct a fundamental solution $\mathcal{E}(x,t)$ of the Dirac operator, i.e., a solution of the equation

$$(\partial_t+\alpha\cdot\nabla+i\beta m)\,\mathcal{E}(x,t)=\delta(x,t)I, \quad \mathcal{E}(x,t)=0 \quad \text{for} \quad t<0,$$

of the form

$$\mathcal{E}(x,t)=(\partial_t-\alpha\cdot\nabla-i\beta m)\,E(x,t), \tag{3.1}$$

where $E(x,t)\equiv E_t(x)$ is a fundamental solution for the Klein-Gordon operator $(\partial_t^2-\triangle+m^2)$, and E vanishes for $t<0$.

Remark 3.1 *The function $E_t(x)$ is given by*

$$E_t(x)=F^{-1}_{k\to x}\frac{\sin\omega t}{\omega}, \quad \omega\equiv\omega(k)\equiv\sqrt{|k|^2+m^2}. \tag{3.2}$$

Then, by the Paley-Wiener Theorem (see, e.g., [6, Theorem II.2.5.1]), the function of $E_t(\cdot)$ is supported by the ball $|x|\le t$.

Denote by $U(t)$, $t\in\mathbb{R}$, the dynamical group for problem (1.1). Then $U(t)$ is a convolution operator given by

$$\psi(x,t)=U(t)\psi_0=\mathcal{E}(\cdot,t)*\psi_0=(\partial_t-\alpha\cdot\nabla-i\beta m)\,E_t(\cdot)*\psi_0. \tag{3.3}$$

The convolution exists because the distribution $\mathcal{E}(\cdot,t)$ is compactly supported by (3.1) and by Remark 3.1.

3.2 Local estimates

Proposition 3.2 *For every* $\psi_0 \in \mathcal{H}$ *and* $R > 0$ *,*

$$\|U(t)\psi_0\|_{0,R} \le C\|\psi_0\|_{0,R+t}, \quad t \in \mathbb{R}, \tag{3.4}$$

where $C < \infty$ *does not depend on* R *and* t *.*

Proof. In the Fourier transform, the solution $\psi(x,t)$ of the Cauchy problem (1.1) reads as

$$\hat\psi(k,t) = \hat{\mathcal{E}}(k,t)\hat\psi_0(k) = \Big[\cos\omega t - (\alpha\cdot(-ik) + i\beta m)\frac{\sin\omega t}{\omega}\Big]\hat\psi_0(k)$$

by (3.1) and (3.2). Then, for $\psi_0 \in L^2$,

$$\|\psi(\cdot,t)\|_{L^2} = \|\hat\psi(\cdot,t)\|_{L^2} \le C\|\hat\psi_0(\cdot)\|_{L^2} = C\|\psi_0(\cdot)\|_{L^2}. \tag{3.5}$$

Let us consider $\psi_0 \in \mathcal{H}$. Introduce the function $\psi_0^*(x)$ equal to $\psi_0(x)$ for $|x| \le R+t$ and to 0 otherwise. Denote by $\psi(x,t)$ (by $\psi^*(x,t)$) the solution of the Cauchy problem (1.1) with the initial data $\psi_0(x)$ ($\psi_0^*(x)$, respectively). Note that $\psi(x,t) = \psi^*(x,t)$ for $|x| \le R$. Therefore, relation (3.5) implies

$$\|\psi(\cdot,t)\|_R = \|\psi^*(\cdot,t)\|_R \le C\|\psi_0^*(\cdot)\|_{L^2} = C\|\psi_0(\cdot)\|_{R+t}. \qquad \Box$$

3.3 Proof of compactness

Proposition 2.7 follows from the estimate (3.3) below by using the Prokhorov Theorem [13, Lemma II.3.1], as in the proof of [13, Thm. XII.5.2].

Proposition 3.3 *Let the conditions of Theorem A hold. Then, for any positive* R *, there exists a constant* $C(R) > 0$ *such that*

$$\sup_{t\ge 0} E\|U(t)\psi_0\|^2_{0,R} \le C(R) < \infty. \tag{3.6}$$

Proof. Let us write

$$e_t(x) := E|\psi(x,t)|^2, \quad x \in \mathbb{R}^3. \tag{3.7}$$

The mathematical expectation is finite for almost every x by (3.4) and by the Fubini theorem. Moreover, $e_t(x) = e_t$ for almost every $x \in \mathbb{R}^3$ by Condition **S1**. Hence, it follows from the Fubini theorem, (3.4) and Condition **S2** that

$$E\|U(t)\psi_0\|^2_{0,R} \equiv e_t|B_R| \le CE\|\psi_0\|^2_{0,R+|t|} \equiv Ce_0|B_{R+|t|}|, \quad t \in \mathbb{R}. \tag{3.8}$$

Here B_R is the ball $|x| \le R$ in $\mathbb{R}^3$, and $|B_R|$ is the volume of this ball. As $R \to \infty$, we see from (3.8) that $e_t \le Ce_0$. Thus,

$$E\|U(t)\psi_0\|^2_{0,R} = e_t|B_R| \le Ce_0|B_R| < \infty. \qquad \Box$$

4 Convergence of correlation functions

We prove the convergence of the correlation functions for the measures μ_t. This implies Proposition 2.8 in the case of Gaussian measures μ_0. It follows from condition **S1** that

$$Q_t^{ij}(x,y) = q_t^{ij}(x-y), \quad x,y \in \mathbb{R}^3, \tag{4.1}$$

for $i,j = 1,\ldots,8$.

Proposition 4.1 *The correlation functions $q_t^{ij}(z)$, $i,j = 1,\ldots,8$, converge for any $z \in \mathbb{R}^3$,*

$$q_t^{ij}(z) \to q_\infty^{ij}(z), \quad t \to \infty, \tag{4.2}$$

where the functions $q_\infty^{ij}(z)$ are defined in (2.17).

Proof. Using the notation (2.14) and (2.15), by (3.3) we obtain

$$\mathcal{R}\psi(x,t) = \Big(\partial_t - P(\nabla)\Big)E_t * \mathcal{R}\psi_0.$$

Then, by (3.2) and (2.15), the Fourier transform of the solution to Cauchy problem (1.1) becomes

$$\widehat{\mathcal{R}\psi}(k,t) = \hat{\mathcal{G}}_t(k)\widehat{\mathcal{R}\psi_0}(k), \quad \text{where } \hat{\mathcal{G}}_t(k) := \cos\omega t - P(-ik)\frac{\sin\omega t}{\omega}. \tag{4.3}$$

The translation invariance (2.10) implies that

$$E(\widehat{\mathcal{R}\psi_0}(k) \otimes \widehat{\mathcal{R}\psi_0}(k')) = F_{x\to k, y\to k'} q_0(x-y) = (2\pi)^3\delta(k+k')\hat{q}_0(k). \tag{4.4}$$

Further, (4.3) gives

$$E(\widehat{\mathcal{R}\psi}(k,t) \otimes \widehat{\mathcal{R}\psi}(k',t)) = (2\pi)^3\delta(k+k')\hat{\mathcal{G}}_t(k)\hat{q}_0(k)\hat{\mathcal{G}}_t^*(k). \tag{4.5}$$

Therefore, by the inverse Fourier transform we obtain

$$\begin{aligned}
q_t(x-y) &= E(\mathcal{R}\psi(x,t)\otimes\mathcal{R}\psi(y,t)) = F^{-1}_{k\to(x-y)}\hat{\mathcal{G}}_t(k)\hat{q}_0(k)\hat{\mathcal{G}}_t^*(k)\\
&= (2\pi)^{-3}\int e^{-ik(x-y)}\Big(\cos\omega t - P(-ik)\frac{\sin\omega t}{\omega}\Big)\hat{q}_0(k)\Big(\cos\omega t - P^T(ik)\frac{\sin\omega t}{\omega}\Big)\,dk\\
&= (2\pi)^{-3}\int e^{-ik(x-y)}\Big[\frac{1+\cos 2\omega t}{2}\hat{q}_0(k) - \frac{\sin 2\omega t}{2\omega}\Big(\hat{q}_0(k)P^T(ik) + P(-ik)\hat{q}_0(k)\Big)\\
&\quad + \frac{1-\cos 2\omega t}{2\omega^2}P(-ik)\hat{q}_0(k)P^T(ik)\Big]\,dk.
\end{aligned} \tag{4.6}$$

To prove (4.2), it remains to show that the oscillatory integrals in (4.6) converge to zero. Let us first analyze the entries of the matrix q_0^{ij}, $i,j = 1,...,8$.

Lemma 4.2 *Let the assumptions of Theorem A hold. Then $\hat{q}_0^{ij} \in L^1(\mathbb{R}^3)$ for any i,j.*

Proof. Let us first prove that

$$q_0^{ij}(z) \in L^p(\mathbb{R}^3), \quad p \geq 1, \quad i,j = 1,...,8. \tag{4.7}$$

Conditions **S0**, **S2** and **S3** imply by (cf. [9, Lemma 17.2.3]) that

$$|q_0^{ij}(z)| \leq C e_0 \varphi^{1/2}(|z|), \quad z \in \mathbb{R}^3, \quad i,j = 1,\ldots,8. \tag{4.8}$$

The mixing coefficient φ is bounded, and hence relations (4.8) and (2.11) imply (4.7),

$$\int_{\mathbb{R}^3} |q_0^{ij}(z)|^p \, dz \leq C e_0^p \int_{\mathbb{R}^3} \varphi^{p/2}(|z|) \, dz \leq C_1 \int_0^\infty r^2 \varphi^{1/2}(r) \, dr < \infty.$$

By Bohner's theorem, $\hat{q}_0^{ij}$ is a nonnegative matrix-valued measure on $\mathbb{R}^3$, and condition **S2** implies that the total measure $\hat{q}_0(\mathbb{R}^3)$ is finite. On the other hand, relation (4.7) for $p = 2$ gives $\hat{q}_0^{ij} \in L^2(\mathbb{R}^3)$. Hence, $\hat{q}_0^{ij} \in L^1(\mathbb{R}^3)$. □

Let us apply this lemma to the oscillatory integrals entering (4.6). The convergence (4.2) follows from (4.6) by the Lebesgue-Riemann theorem. This completes the proof of Proposition 4.1. □

Relation (4.7) for $p = 1$ implies now that $\hat{q}_0(k)$ is bounded. Hence, the explicit formula (2.17) implies the following assertion.

Corollary 4.3 *All matrix elements* $\hat{q}_\infty^{ij}(k)$, $i,j = 1,\ldots,8$, *are bounded.*

5 Convergence of characteristic functionals

To prove Proposition 2.8 for the general case of a non-Gaussian measure μ_0, we develop a version of Bernstein's "room - corridor" method of [1]-[4]: (i) we use an integral representation for the solutions of (1.1), (ii) divide the region of the integration into "rooms" and "corridors" and (iii) evaluate their contribution. As the result, the value $\langle U(t)\psi_0, \phi\rangle$ for $\phi \in \mathcal{D}$ is represented as the sum of weakly dependent random variables. Then we apply Bernstein's "rooms-corridor" method and the Lindeberg central limit theorem.

(i) We first evaluate the inner product $\langle U(t)\psi_0, \phi\rangle$ in (2.19) by using duality arguments. For $t \in \mathbb{R}$, introduce a "formal adjoint" operators $U'(t)$ from the space $\mathcal{D}$ to a suitable space of distributions. For example,

$$\langle \psi, U'(t)\phi\rangle = \langle U(t)\psi, \phi\rangle, \quad \phi \in \mathcal{D}, \quad \psi \in \mathcal{H}. \tag{5.1}$$

Write $\phi(\cdot, t) = U'(t)\phi$. Then (5.1) can be represented as

$$\langle \psi(t), \phi\rangle = \langle \psi_0, \phi(\cdot, t)\rangle, \quad t \in \mathbb{R}. \tag{5.2}$$

The adjoint groups admit a convenient description (see Lemma 5.1 for the group $U'(t)$).

Lemma 5.1 *For* $\phi \in \mathcal{D}$, *the function* $U'(t)\phi = \phi(x,t)$ *is the solution of*

$$\dot{\phi}(x,t) = (\alpha \cdot \nabla + i\beta m)\phi(x,t), \quad \phi(x,0) = \phi(x). \tag{5.3}$$

Proof. Differentiating (5.1) with respect to t for $\psi, \phi \in \mathcal{D}$, we obtain

$$\langle \psi, \dot{U}'(t)\phi\rangle = \langle \dot{U}(t)\psi, \phi\rangle. \tag{5.4}$$

The group $U(t)$ has the generator $\mathcal{A} = -\alpha \cdot \nabla - i\beta m$. Therefore, the generator of $U'(t)$ is the conjugate operator

$$\mathcal{A}' = \alpha \cdot \nabla + i\beta m. \tag{5.5}$$

Hence, relation (5.3) holds indeed with $\dot{\phi} = \mathcal{A}'\phi$. □

Remark 5.2 *Comparing (5.3) and (1.1), we see that $\phi(x,t) = U'(t)\phi$ can be represented as a convolution (cf (3.3)), namely,*

$$\phi(\cdot, t) = \mathcal{R}_t * \phi, \quad \mathcal{R}_t := (\partial_t + \alpha \cdot \nabla + im\beta)E_t. \tag{5.6}$$

(ii) Introduce a "room-corridor" partition of $\mathbb{R}^3$. For a given $t > 0$, choose $d_t \geq 1$ and $\rho_t > 0$ such that $\rho_t \sim t^{1-\delta}$ with some $\delta \in (0,1)$ and $d_t \sim t/\ln t$, as $t \to \infty$. Set $h_t = d_t + \rho_t$ and

$$a^j = jh_t, \;\; b^j = a^j + d_t, \;\; j \in \mathbb{Z}. \tag{5.7}$$

We refer to the slabs $R_t^j = \{x \in \mathbb{R}^3 : \; a^j \leq x^3 \leq b^j\}$ as "*rooms*" and to $C_t^j = \{x \in \mathbb{R}^3 : \; b^j \leq x^3 \leq a_{j+1}\}$ as "*corridors*". Here $x = (x^1, x^2, x^3)$, the symbol d_t stands for the *width* of a room, and ρ_t for that of a corridor.

Denote by χ_r the indicator of the interval $[0, d_t]$ and by χ_c that of $[d_t, h_t]$, which means that

$$\sum_{j \in \mathbf{Z}} (\chi_r(s - jh) + \chi_c(s - jh)) = 1 \quad \text{for (almost all)} \;\; s \in \mathbb{R}.$$

The following decomposition holds:

$$\langle \psi_0, \phi(\cdot, t)\rangle = \sum_{j \in \mathbf{Z}} (\langle \psi_0, \chi_r^j \phi(\cdot, t)\rangle + \langle \psi_0, \chi_c^j \phi(\cdot, t)\rangle), \tag{5.8}$$

where $\chi_r^j := \chi_r(x^3 - jh)$ and $\chi_c^j := \chi_c(x^3 - jh)$. Consider the random variables r_t^j and c_t^j given by

$$r_t^j = \langle \psi_0, \chi_r^j \phi(\cdot, t)\rangle, \;\; c_t^j = \langle \psi_0, \chi_c^j \phi(\cdot, t)\rangle, \qquad j \in \mathbb{Z}. \tag{5.9}$$

Then (5.8) and (5.2) imply

$$\langle U(t)\psi_0, \phi\rangle = \sum_{j \in \mathbf{Z}} (r_t^j + c_t^j). \tag{5.10}$$

The series in (5.10) is in fact a finite sum. Indeed, for the support of ϕ we have

$$\operatorname{supp} \phi \subset B_{\overline{r}} \quad \text{ for some } \; \overline{r} > 0.$$

Then, by the convolution representation (5.6), the support of the function $\phi(\cdot, t)$ at $t > 0$ is a subset of an "inflated future cone"

$$\operatorname{supp} \phi \subset \{(x,t) \in \mathbb{R}^3 \times \mathbb{R}_+ : \; |x| \leq t + \overline{r}\}, \tag{5.11}$$

whereas $\mathcal{R}_t(x)$ is supported by the 'future cone' $|x| \leq t$. The latter fact follows from (5.6) and from Remark 3.1. Finally, it follows from (5.9) that

$$r_t^j = c_t^j = 0 \quad \text{ for } \quad jh_t + t < -\overline{r} \quad \text{ and for } \quad jh_t - t > \overline{r}. \tag{5.12}$$

Therefore, the series (5.10) becomes a sum,

$$\langle U(t)\psi_0, \phi\rangle = \sum_{-N_t}^{N_t} (r_t^j + c_t^j), \quad N_t \sim \frac{t}{h_t}. \tag{5.13}$$

Lemma 5.3 *Let Conditions* **S0–S3** *hold. Then the following bounds hold for* $t > 1$ *:*

$$E|r_t^j|^2 \le C(\phi)\ d_t/t, \quad E|c_t^j|^2 \le C(\phi)\ \rho_t/t, \quad j \in \mathbb{Z}. \tag{5.14}$$

Proof. We discuss the first bound in (5.14) only, because the other can be proved in a similar way. Rewrite the LHS of (5.14) as the integral of correlation functions. We obtain

$$E|r_t^j|^2 = \langle \chi_r^j(x_3)\chi_r^j(y_3) q_0(x-y), \phi(x,t)\otimes\phi(y,t)\rangle. \tag{5.15}$$

The following uniform bound holds (cf. [11, Thm. XI.17 (b)]):

$$\sup_{x\in\mathbb{R}^3} |\phi(x,t)| = \mathcal{O}(t^{-3/2}), \quad t\to\infty. \tag{5.16}$$

In fact, (5.6) and (3.2) imply that the function $\phi(x,t)$ can be represented written as the sum

$$\phi(x,t) = \sum_{\pm} \int_{\mathbb{R}^3} e^{-i(kx\pm\omega t)} a^{\pm}(\omega)\hat\phi(k)\ dk, \tag{5.17}$$

where $a^{\pm}(\omega)$ is a matrix whose entries are linear functions of ω or $1/\omega$. Let us prove the asymptotics (5.16) along each ray $x = vt + x_0$ with $|v| \le 1$. The asymptotic relation thus obtained must hold uniformly in $x \in \mathbb{R}^3$ by (5.11). By (5.17) we have

$$\phi(vt + x_0, t) = \sum_{\pm} \int_{\mathbb{R}^3} e^{-i(kv\pm\omega)t - ikx_0} a^{\pm}(\omega)\hat\phi(k)\ dk. \tag{5.18}$$

This is a sum of oscillatory integrals with the phase functions $\phi_{\pm}(k) = kv \pm \omega(k)$. Each function has two stationary points which are solutions of the equation $v = \mp\nabla\omega(k)$ if $|v| < 1$, and has none if $|v| \ge 1$. The phase functions are nondegenerate, i.e.,

$$\det\left(\frac{\partial^2\phi_{\pm}(k)}{\partial k_i \partial k_j}\right)_{i,j=1}^{3} \ne 0, \quad k \in \mathbb{R}^3. \tag{5.19}$$

Finally, $\hat\phi(k)$ is smooth and rapidly decays at infinity. Therefore, $\phi(vt + x_0, t) = \mathcal{O}(t^{-3/2})$ according to the standard method of stationary phase, see [7].

According to (5.11) and (5.16), it follows from (5.15) that

$$E|r_t^j|^2 \le Ct^{-3} \int_{|x|\le t+\overline{r}} \chi_r^j(x^3)\|q_0(x-y)\|\ dxdy = Ct^{-3} \int_{|x|\le t+\overline{r}} \chi_r^j(x^3)dx \int_{\mathbb{R}^3} \|q_0(z)\|dz, \tag{5.20}$$

where $\|q_0(z)\|$ stands for the norm of the matrix $\left(q_0^{ij}(z)\right)$. Therefore, relation (5.14) follows for $\|q_0(\cdot)\| \in L^1(\mathbb{R}^3)$ by (4.7). $\square$

Hence, the rest of the proof of Proposition 2.8 is the same as that in the case of the Klein-Gordon equation, [1, p.20-25]. The proof of Theorem A is complete.

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